

h is classical

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Regardless of number, standing wave modes are by definition noninteracting, and therefore cannot thermalize by themselves. Doppler shifts due to thermal motions of cavity walls provide necessary mixing, but also preserve the amplitudes and phases. The $\lambda/2$ intervals of the modes thus preserved must have equal energy expectations, say $\langle \epsilon \rangle$, in the resulting equilibrium. By definition again, they can be exchanged between modes only in whole numbers and hence only between harmonics. Each family of harmonic modes is thus self-contained and is disjoint from other families in such exchanges, and further, can have no more than one mode excited at any instant. The second property identifies harmonic families of standing wave modes as the harmonic oscillators of Planck's theory, since a family can only bear energy equal to exactly one of ϵ , 2ϵ , 3ϵ , etc. These two properties further imply that the energy expectation gets averaged only over an individual family, as the equilibrium energy $\langle E(\nu) \rangle$ steadily available at a given mode would have contributions from its entire harmonic family. Planck's equations reemerge, and radiation quantization arises as a classical rule $\langle E(\nu) \rangle = \langle \epsilon \rangle \nu$, as a mode must contain a whole number of exchangeable $\lambda/2$ intervals, but it only concerns equilibrium states. The result makes Planck constant h an analogue of Boltzmann's constant k_B for the frequency domain, and points to postulate-free explanations of all aspects of quantum and kinetic theories.

I. INTRODUCTION

Presented below is a first-ever analysis of the equilibrium of standing wave modes themselves subject to irreversible interactions with the environment via the thermal motions of confining walls. The wall interactions are provided by the classical Doppler shifts applicable to all waves and all wavelengths, regardless of the molecular, atomic or subatomic properties of wall matter, and overcome Boltzmann's argument of 1897 that Maxwell's equations are symmetric and do not provide irreversibility [1], obviating Planck's hypothesis of "elementary disorder" [*ibid.*] for the determination of the equilibrium spectral distribution. Two basic aspects of this dynamic equilibrium are established, as follows.

- The half-wavelength ($\lambda/2$) intervals of the standing wave modes would be statistically preserved in number at all times, thus providing the exact analogues to "atoms" in the kinetic theory sought by Planck. More particularly, the $\lambda/2$ intervals would, in effect, be exchanged between standing wave modes only in whole numbers.

Doppler shifts preserve both the amplitude and the phase profiles of reflected wave trains exactly, hence the $\lambda/2$ intervals would be atomic under this mechanism, but the preceding arguments are statistical and independent of the precise mechanisms responsible for the mixing of the mode energies. Absorption and emission by matter are still necessary for complete relaxation, which should include splitting and recombination of wave trains but irreversibility *per se* is adequately provided by the wall shifts and does not depend on relaxation. This seems to resolve another long pending issue in the kinetic theory, known as the Fermi-Pasta-Ulam (FPU) problem, that large number simulation models continue to display periodicities uncharacteristic of heat [2] [3, §5.5.1] – for real systems, the notion of equilibrium invariably includes equilibrium with the environment.

- The combination of relaxation and irreversible mixing should drive the confined radiation to a state in which all of the entities preserved by the mixing, viz the $\lambda/2$ intervals, would have equal expectation energies. This incidentally means that the equalized energy integrals are relative to phase rather than physical space. Further, these expectation energies must match those of similar atomic entities outside the cavity walls, and must thus be universal. This, then, is the true equivalent of the equipartition in kinetic theory for radiation.

Since the number of $\lambda/2$ intervals in a standing wave mode is proportional to its frequency, Planck's quantization rule $E(\nu) = h\nu$ is directly reproduced classically in this state of dynamic equilibrium. Two further observations also follow from the definition of standing wave modes, viz that the $\lambda/2$ intervals can be exchanged only between harmonically related modes, and that every such exchange would destroy the excitation in the originating mode while populating the destination mode. These further observations suffice, as will be shown, to identify Planck's oscillators with real families of harmonically related modes, and to reproduce Planck's law *classically*.

The present result demolishes a century old conclusion, based on an analysis by Rayleigh and Jeans and informally reinforced by Einstein's photoelectricity theory, that quantization is *absolutely underivable* from classical physics (cf. [4, I-41-2]). Rayleigh and Jeans attributed the equipartition energy of $k_B T/2$, established for particulate entities in the kinetic theory, to individual standing wave modes, and this famously led to the problem of "ultraviolet divergence". However, their more basic error of attributing the *entire* time-domain equipartition energy to *each* Fourier component in the same equilibrium, has gone unnoticed – this subsumes the Rayleigh-Jeans divergence and questions the precise

use of Fourier decomposition in the subsequent quantum theory. Persisting speculations like “Hamilton had no basis to assume that h was anything but zero” [5, §10-8], are a case in point, since Bohr’s Correspondence Principle theory [*ibid.*] makes h the scale factor for the frequency domain – and $h = 0$, the only logically inadmissible value!

More generally, individual Fourier components are notionally identified with particles in current physics, explaining, in hindsight, why the Boltzmann statistics, which leads to the time domain equipartition of $k_B T/2$, has to be replaced in quantum theory by the boson or fermion statistics applicable specifically, and only, to the Fourier components. This association of Fourier components with particulate behaviour is only partly explained by Hamilton-Jacobi theory and the Correspondence Principle, which leave the very nature of quantum wavefunctions, the mechanism of “wavefunction collapse”, and so on, to as yet speculative interpretations. A complete theory must include a very general treatment of the physical processes of observation to actually explain the origin of quantization, i.e. mundanely, without indirectly reintroducing equivalent postulates. Such a theory is mandated and indeed made possible by the present result as the separation of irreversibility from relaxation admits irreversible interactions with *the Hamilton-Jacobi standing wave components of the inequilibrium states of the observers and their instruments* that represent information (cf. [6, 7]). This would suffice to fundamentally ensure that the standing wave mode $\lambda/2$ intervals also necessarily characterize *all* observations by *all physical* observers, thus finally proving the universality of quantization on formal classical grounds, limiting its applicability to states of thermalization and reducing the probabilistic nature of wavefunctions to heat.

This broader picture would be incomplete without also noting why a classical foundation is necessary, and sufficient, for a fundamental understanding of nature, apparently contradicting all of the 20th century wisdom viewing subatomic elementary particles as the ultimate constituents of matter. The necessity arises from the fact that real observers and processes of observation must be macroscopic by definition, and therefore classical. More particularly, the observer’s physical states representative of observed data must be also macroscopic for the same reason, as well as stationary to be representative of knowledge. The sufficiency follows from the fact that the macroscopic observer interface can have at most a finite precision, hence any finite combination of observer states necessarily represents at best a continuum of possible configurations of nature that cannot be distinguished by those states. In other words, *the finiteness of real observers prevents them from precisely observing and representing anything but continuous fields*, equivalent to a finite combination of Fourier terms. Moreover, the inherently macroscopic nature of observers makes it moot whether the elementary particles are as fundamental as currently thought, or merely reflections of the energy being thrown into the observational processes in high energy research. The Hamilton-Jacobi standing wave modes of the observer suffice as a basis for representing observable patterns at the observer interface, and would be fundamental in terms of reducibility of the laws of physics to invariants of these patterns, providing a formal foundation for reasoning in physics analogous to the Turing machine in computational science. Classical physics happens to be consistent with this foundation, to the extent that it relies on definitions rather than postulates [19].

Correspondingly, the particulate approach has led to at least two fundamental limitations in modern physics. First, it has obfuscated the very consideration of successive diffractions, i.e. continued bending of already diffracted portions of wavefronts due to further diffraction by other obstructions encountered by these portions. Successive diffractions are well understood in the form of surface effects, for example, in the known capture of passing wavefront energy by a sphere virtually forever, but past considerations of diffraction in astronomy and quantum field theories are limited to Fresnel or Fraunhofer approximations, for dust attenuation [8, pp149-153] and scattering [9, pp149,158], respectively. These approximations only describe total deflections of less than $\pi/2$, whereas the spherical capture alone suffices to change the fundamental wave propagation law from the currently used r^{-2} law for free space to a weakly attenuated form $e^{-\sigma r} r^{-2}$ for both the real, matter-filled universe and inside dense matter like the sun’s interior. Its contributions to Olbers’ paradox, galactic dark matter and neutrino oscillations remain to be verified. A second, more basic wave effect, which was also previously unrecognized but will be hopefully soon verified on laboratory scale, is a Doppler-like scaling of observed frequencies, but in proportion to source distances like Hubble’s law [10, 11, 12], and driven solely by temporal variation of the observing instrument and thus its states [20].

The key contributions are thus a more precise thermodynamic treatment of confined radiation; the correct principle of equipartition for the frequency domain; separation of irreversibility from relaxation solving the FPU problem and improving over Boltzmann’s ideas; and a principle of reality governing every observer of relevance to physics, making it necessary to consider its representative states and leading to universal quantization as explained. The rest of this paper is concerned with the radiation equilibrium, and with proving the following theorem representing its core result.

Theorem 1 (Classical equivalence of quantum mechanics) *Planck’s law implies and is implied by a classical equilibrium of standing wave modes of radiation with their environment.*

The forward implication of a classical equilibrium is proved first in Section II, both to establish the consistency of the classical derivation to be given in Section III, and to provide needed insight into the nature of travelling waves. The main result is of the course the derivation itself, in Section III, proving the reverse implication that the classical equilibrium of standing wave modes yields Planck’s law, with no postulate that might imply inherent quantization. Appendix C explains the Correspondence Principle issue in detail.

II. QUANTIZATION IMPLIES A CLASSICAL EQUIPARTITION

The forward implication of Theorem 1, that Planck's law in fact implies a classical equipartition over $\lambda/2$ intervals of standing wave modes, is proved below as a slightly more general lemma.

Lemma 1 (Spectral equipartition) *For confined radiation in thermal equilibrium, Planck's quantization rule,*

$$E(\nu) = h\nu, \quad (1)$$

implies uniform distribution of energy between full wavelength intervals of arbitrary periodic waveforms, and between half-wavelength intervals of sinusoidal waves.

Proof. As the wave travels through any cross-sectional plane in space, its phase at the plane will vary over a full cycle. Hence, if we integrate over a very small phase interval $\phi\lambda/2\pi$, where $\phi \ll 2\pi$, the instantaneous total energy in the spatial interval starting at the plane will vary between 0 and $a^2 A \mathcal{F}(f, \phi)$, where A is the cross-sectional area chosen and $\mathcal{F}(f, \phi)$ is a shape factor that depends on the waveform f , with $\mathcal{F}(\sin, \pi) = 1/2$.

For the identified interval lengths, viz λ intervals ($\phi = 2\pi$) in general and $\lambda/2$ intervals ($\phi = \pi$) for sinusoids, the total wave energy in such an interval cannot change as the wave passes. This is proved as a lemma in Appendix A. For the sinusoidal intervals, the energy integral *with respect to phase* is (see Fig. 1)

$$U(\pi, \alpha, \sin) \equiv A \int_{\theta=\alpha}^{\alpha+\pi} a^2 \sin^2(\theta) d\theta \equiv A \int_0^\pi a^2 \sin^2(\theta + \alpha) d\theta = A \cdot \pi a^2 / 2 \quad (2)$$

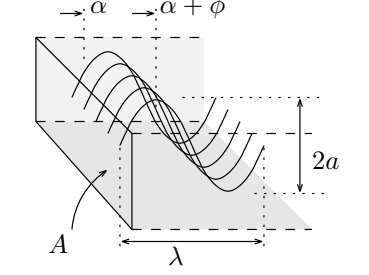


FIG. 1: The integral $U(\phi, \alpha, \sin)$

and occupies a length of $\lambda/2$. It represents an energy density $u \equiv U(\pi, \alpha, \sin)/(A\lambda/2) = \pi a^2/\lambda \equiv \pi a^2\nu/c$.

The length of an interval spanning a phase difference of ϕ is $\lambda \cdot \phi/2\pi \equiv c\phi/2\pi\nu$, where λ is the wavelength, ν , its frequency and c denotes the speed. If the internal dimension of the cavity along the direction of the wave is L , there would be $L \cdot 2\pi\nu/c\phi$ such intervals, and more particularly $2\nu L/c$ of $\lambda/2$ intervals. The total wave energy is then

$$E(\nu) = (A\lambda/2) \cdot u \cdot 2\nu L/c = a^2\pi\nu V/c, \quad (3)$$

where $V \equiv AL$ is the volume, for a rectangular cavity. When the cross-section is uneven, A would need to be integrated along L , but the overall result would be clearly preserved. Dividing equation (1) by this total wave energy yields

$$a^2 = hc/\pi V, \quad (4)$$

which is independent of the frequency. Thus, the quantization rule, equation (1), means that sinusoidal waves will all have the same mean amplitude $|a|$, i.e. a uniform distribution of energy among $\lambda/2$ intervals regardless of λ . The generalization to λ intervals of an arbitrary periodic waveform (Fig. 2) follows from the Fourier series decomposition of such a waveform into a whole number of wavelength intervals of its sinusoidal components. By a lemma proved in Appendix A, which is applicable to these components, the phase offset of the full wavelength interval of the parent waveform cannot matter with respect to its energy.

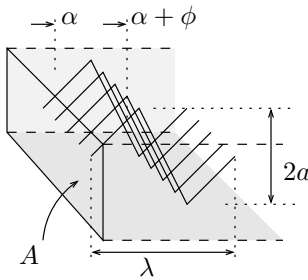


FIG. 2: The integral $U(\phi, \alpha, f)$

The Fourier decomposition is equal to the sum of an infinite series of sinusoidal waves with nodes coinciding with the end points of its period, as

$$f(x) = \sum_{n=0}^{\infty} F_n e^{ik_n x} \quad \text{where } k_n = 2\pi/\lambda_n \text{ and } F_n = \int_{x=0}^{\lambda} f(x) e^{ik_n x} dx. \quad (5)$$

Equation (2) must hold for each of these sinusoidal components, provided that we refer the corresponding phase variables ϕ_i and α_i for each component to the parent waveform. The energy of a full wavelength interval of the parent is then

$$U^*(\phi, \alpha, f) = \sum_{n=0}^{\infty} F_n U(n\phi, n\alpha, \sin). \quad (6)$$

Equality of the energy sums $U^*(2\pi, \alpha, f)$ is then assured regardless of $\alpha \in [0, 2\pi)$, as the condition individually holds for each of the sinusoidal components independently of their starting phases, as remarked. \square

Discussion. This is a more general result than needed for Theorem 1, since it holds for λ -intervals of all waveforms, instead of merely the sinusoidal components constituting the standing wave modes. It suggests a deeper significance of Planck's discovery which has hitherto escaped attention, viz that (half-)wavelength intervals of unit cross-section behave like particulate bearers of energy, similar to the molecules of a gas in the kinetic theory. The equipartition is different from that of kinetic theory, however, since we have *equal amplitude expectations*, instead of the *equal energy expectations*. A naïve application of the classical equipartition would be based on the energy integral *over space*,

$$V(x_1, x_0, \sin) \equiv \int_{x_0}^{x_1} a^2 \sin^2(kx') dx' \equiv \frac{a^2}{k} \int_{\theta=kx_0}^{kx_1} \sin^2(\theta) d\theta \equiv \frac{a^2}{k} \int_{kx'=kx_0}^{kx_1} \sin^2(\theta + kx_0) d\theta \equiv \frac{a^2}{k} U(kx, kx_0, \sin). \quad (7)$$

However, equation (7) is merely the energy density of a wave by itself. It has no bearing on the thermal equilibrium as the classical principle of superposition of electromagnetic fields implies a complete absence of dynamical interaction between their waves. This is in effect why Planck was forced to stipulate his principle of “elementary disorder”, which was interpreted in the Rayleigh-Jeans analysis in terms of absorption and emission by atomic matter [4, I-41-2]. The equal energy approach is inherently flawed, however, when applied to standing wave modes since the latter constitute individual Fourier components of radiation, rather than independent, full-fledged dynamical entities, as remarked. In particular, as the frequency domain is infinite, the idea inherently implied infinite energy.

Lemma 1 favours Doppler shifts as the driving mechanism for thermalization, since they are the only mechanism for energy transfer between standing wave modes guaranteed to preserve amplitudes and phase intervals.

III. CLASSICAL EQUILIBRIUM YIELDS QUANTIZATION

The reverse implication of Planck's law and the radiation quantization from the classical equilibrium of the standing wave modes requires showing first that the standing wave modes are as such a complete set of interacting entities for describing the radiation equilibrium, and then showing that Planck's law results from their classical equilibrium.

The equilibrium is considered to be of the modes themselves, which closely relates to the Bose-Einstein derivation and second quantization picture, instead of a mere balance of power flows in interactions with matter as considered in the Rayleigh-Jeans analysis. The key difference from both Planck's and the Bose-Einstein derivations is, of course, that these interacting entities are real and mundane, in existence as well as properties, rather than postulated entities with hypothesized behaviour that happened to yield the same spectral distribution.

This more substantive role of standing wave modes, of actually characterizing the equilibrium distribution of radiant energy by themselves with no more than perfectly classical interactions, had thus remained unobvious, likely thanks to distraction by the notion of quantization. It is accordingly established first as a lemma below. The rest of the proof of Theorem 1 follows from Lemma 3, deriving Planck's law from this classical equilibrium. The universal nature of h follows from the form of Planck's law, just as in traditional theory, and is included as a corollary for reference.

Lemma 2 (Sufficiency of standing wave modes) *Standing wave modes constitute a complete set for representing the equilibrium energy distribution of confined radiation.*

Proof. The total energy U in the cavity is instantaneously comprises three parts:

- a portion U_0 contained in the standing wave modes,
- a portion U_- contained in travelling waves that happen to be losing energy to other frequencies due to the wall Doppler shifts (Fig. 3), and
- a similar travelling wave portion U_+ gaining energy from other frequencies.

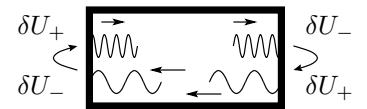


FIG. 3: Mixing at the walls

We have $dU_0/dt = 0$ by definition, hence statistical constancy of the total energy in equilibrium means that $dU_-/dt = -dU_+/dt$. Any occurrence of U_- and U_+ components signifies mixing of wave energies across frequencies. The mixing must occur due to absorption and emission by matter comprising the confining walls or contained within the cavity, as well as by Doppler shifts from the thermal motions of the walls. The wall motions are implied by the very premise of equilibrium even in Planck's theory, because for equilibrium at a temperature T , the cavity must be kept in thermal contact with a reservoir at T .

As remarked in the Introduction, the wall Doppler shifts cannot account for the relaxation of any *a priori* excited standing wave modes as Doppler shifts preserve both amplitudes and phase extents of reflected wave trains. They do suffice to introduce irreversibility, however, and to effect mixing of mode energies at a potentially greater rate limited only by the wave speed within the cavity, separately from the relaxation due to absorption and emission by matter. The precise rates are not important for the equilibrium distribution, so long as they are nonzero.

Consider first the hypothetical ideal conditions of perfectly rigid, non-interacting walls that neither absorb nor emit energy, and must therefore perfectly reflect all incident radiant energy. This was implicitly assumed in Planck's theory and also in the Rayleigh-Jeans analysis, as both ignored the possible thermalizing effects of wall motions. Incidentally, this has also been a subtle limitation in the kinetic theory of gases, where the notion of entropy given by Boltzmann's H -theorem has been historically attributed to statistical probabilities of large numbers alone, but this has been shown by simulations to be inadequate, as mentioned.

In the present context of confined radiation, for every incident travelling wave $w_{(-)}$, there must then be an outgoing reflected travelling wave $w_{(+)}$ of exactly the same frequency and amplitude, since there would be no shift of frequency and the instantaneous power $dU(w_{-})/dt$ of the incident wave must be exactly balanced by the power $dU(w_{+})/dt$ of the reflected wave. The incident and reflected waves therefore form standing waves, since the condition amounts to $dU(w_{-})/dt + dU(w_{+})/dt = 0$ for each $w_{(-)}$. These idealized conditions thus permit only standing waves, hence the standing wave modes necessarily comprise the complete energy spectrum of the radiation.

In the real world scenario where the walls are not perfectly reflecting, that is, where atomic interactions and thermal Doppler shifts are involved, there would be a mean outflow dU_{-}/dt of energy going into the walls and atoms therein, and a statistically equal inflow dU_{+}/dt entering the cavity, for the statistical constancy of the total energy U .

We cannot assume *a priori* that the instantaneous values of dU_{-}/dt and dU_{+}/dt will also match at every frequency, as imperfect reflectivity means that some frequencies will likely be gaining energy, but the gain must eventually occur at the expense of others for the statistical constancy of U over an extended period of observation. Instantaneously, there will be travelling waves going into the walls and losing energy at some frequencies as the outbound waves happen to be absent or weaker at these frequencies, and likewise, there will be waves emanating from the walls and the atoms therein at other frequencies with weaker or no corresponding inbound waves. Over a sufficiently long interval of time, there should be no net gain or loss at any individual frequency ν .

Since the expectation value $E(\nu) = \langle U(\nu) \rangle$ is defined by averaging over all such waves over a long enough period of observation (ideally infinite), it would be quantitatively equal to the average computed when no gain or loss occurred at all at any frequency ν over that same interval. This is the same as assuming that all of the energy at frequency ν had been contained solely in standing wave modes. \square

Classical equipartition of radiation. According to classical statistical mechanics, in the state of equilibrium of any system of interacting entities, such as gas molecules in the kinetic theory, there is equal probability of finding any individual entity at energy u' , given by the probability density function

$$p \equiv p(u') \propto e^{-u'/k_B T}. \quad (8)$$

We would expect equation (8) to be identically applicable for the radiation equilibrium in terms of any set of energy-bearing components of the radiation in the cavity that, like the molecules of a gas, will ordinarily be neither created nor destroyed but can possess different energies.

The premise of equilibrium at a temperature $T > 0$ as such implies equilibrium with the immediate environment also at T , and this implies involvement of wall Doppler shifts as a key, classical mechanism of irreversibility, as mentioned in the Introduction. It has the remarkable property of also conserving the number of $\lambda/2$ intervals of both travelling and standing waves, but according to the preceding lemma, the latter contain all of the radiant energy in equilibrium. To be precise, the Doppler shifts only suffice to translate any standing wave excitation to various other frequencies in a random sequence. Absorption and emission by matter is necessary for dispersing individual excitations to multiple, concurrent wave trains, and would break the conservation of $\lambda/2$ intervals during relaxation. In equilibrium, however, these relaxation transformations would have already arrived at a steady state, so that the conservation of $\lambda/2$ intervals would be still assured in a statistical sense. Equation (8) should be therefore applicable to the sinusoidal $\lambda/2$ intervals of unit cross-sectional area for which

$$u' \sim u_\pi \equiv U(\pi, \alpha, \sin)/A = \pi a^2/2. \quad (9)$$

It follows that in the classical equipartition, all $\lambda/2$ intervals would exhibit the same expectation energy u' with the probability $p(u')$ given by equation (8). This suffices to yield Planck's law, as asserted by the following lemma.

Lemma 3 (Classical distribution) *A Boltzmann distribution of the half-wavelength intervals of sinusoidal standing wave modes yields Planck's law.*

Proof. By the preceding discussion, this result represents radiation equilibrium in general, inclusive of all contributing mechanisms, even relaxation processes that do not preserve the number of the $\lambda/2$ intervals. Only rectangular cavities need to be considered, as the expectation energy distribution of an irregular cavity in equilibrium with a rectangular cavity must match the latter's distribution at each frequency, by the principle of detailed balance.

The two component aspects of most interest would be the reproduction of Planck's quantization rule, equation (1), and the identification of classical radiative entities exhibiting the very properties required by Planck's hypothesis of quantized harmonic oscillators. The first is readily obtained by applying Section II in reverse to the equipartition just described, since a standing wave mode of frequency ν must geometrically comprise $2\nu L/c$ of $\lambda/2$ intervals (equation 3), so that the instantaneous mode energy, for unit cross-sectional area, is given by

$$E(\nu) = \frac{2\nu Lu'}{c} \propto \nu u', \quad (10)$$

though with the Boltzmann probability

$$p(\nu) = p^{2\nu L/c}, \quad (11)$$

given by equation (8) applicable to each $\lambda/2$ interval within the mode.

Planck's harmonic oscillators are given, as stated in the Introduction, by *harmonic families*, i.e. families (or sets) of harmonically related standing wave modes, $H\{\nu_0\} \sim \{\nu_0, 2\nu_0, 3\nu_0, \dots\}$, for any chosen mode of frequency ν_0 . We need only the following two properties for this identification to hold:

- a harmonic oscillator may oscillate at any of a discrete set of harmonically related frequencies and no other; and
- a harmonic oscillator can be at only one frequency ν , or equivalently, at one energy level $E = h\nu$, at any instant.

The first property already follows from the definition of the harmonic families. The second property is not so trivial, as multiple members of a harmonic family ordinarily can be in excited states with various energy levels simultaneously. However, the equilibrium state statistically preserves $\lambda/2$ intervals, hence the equilibrium interactions are equivalent to exchanges only of whole numbers of $\lambda/2$ intervals between standing wave modes. Further, the exchanges happen to be impossible, by the geometrical nature of standing wave modes, unless they are strictly between members of the same harmonic family. A single interval exchange between two modes, possessing m and n $\lambda/2$ -intervals, respectively, means the pair of transitions

$$\left. \begin{array}{l} [m] \rightarrow [m+1] \\ [n] \rightarrow [n-1] \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} [m] \rightarrow [m-1] \\ [n] \rightarrow [n+1] \end{array} \right.,$$

where $[r]$ denotes a standing wave mode of r whole $\lambda/2$ intervals, but each such interacting pair $[m]$ and $[n]$ represent m -th and n -th harmonics of a fundamental mode that would be denoted $[1]$. *The state of equilibrium thus represents a disjoint partitioning of both standing wave modes and their interactions into harmonic families.* The second property, that each family acts as if it has no more than one active member at any time, follows upon noting that each of these exchange interactions logically destroys the original pair of standing waves. Equation (11) implies $p([n]) = p^n$.

Equation (10) means simply that measurements of a standing wave mode of frequency ν will yield the energy value $E(\nu)$ only with the probability $p(\nu)$ given by equation (11). The equilibrium energy spectrum requires the expectation value $\langle E(\nu) \rangle$, which has to be computed by averaging the energies in all modes, weighted by their probabilities, that can interact with the modes at frequency ν . By the preceding arguments of the equilibrium state and Doppler mixing, this means all modes of frequency ν and their harmonics, since any of the harmonic waves can contribute one or more of their $\lambda/2$ intervals to the measured modes.

The averaging is only necessary over a single harmonic family, because other harmonic families, belonging to other polarizations, directions or propagation paths, are effectively noninteracting, and will independently yield a similar distribution. Their contribution can be accounted for by incorporating a density of modes factor, which amounts to $8\pi\nu^2/c^3$, as reproduced in Appendix B from basic texts for reference. Further, the averaging needs to be only over the harmonic family of the immediately measured frequency ν , and excluding even its subharmonics, because while subharmonic standing waves can contribute power at the measured frequency by supplying $\lambda/2$ intervals, the entire measured mode cannot be replenished from one of its subharmonics. This corresponds to the equivalent equilibrium distribution of the standing wave modes instead of simply the $\lambda/2$ intervals, and in this picture, the full set of $2\nu L/c$ $\lambda/2$ -intervals of a mode must be exchanged as a whole. The sum of energy densities in the harmonic family of ν is

$$E(H\{\nu\}) = u_\pi \times [\nu \cdot p^\nu + 2\nu \cdot p^{2\nu} + 3\nu \cdot p^{3\nu} + \dots] = \frac{u_\pi \nu \cdot p^\nu}{(1 - p^\nu)^2}, \quad (12)$$

and this is spread over a total of

$$N(H\{\nu\}) = 1 + p^\nu + p^{2\nu} + p^{3\nu} + \dots = \frac{1}{1 - p^\nu} \quad (13)$$

modes in the harmonic family, hence the expectation energy density at ν , neglecting the density of modes, is

$$\langle u_\pi(\nu) \rangle \equiv \frac{E(H\{\nu\})}{N(H\{\nu\})} = \frac{u_\pi \nu}{p^{-\nu} - 1} = \frac{u_\pi \nu}{e^{u_\pi \nu / k_B T} - 1}. \quad (14)$$

With the density of modes, the spectral energy density becomes

$$\langle E(\nu, T) d\nu \rangle = \frac{8\pi\nu^2}{c^3} \frac{u_\pi \nu d\nu}{e^{u_\pi \nu / k_B T} - 1}. \quad (15)$$

This is Planck's law, but with the $\lambda/2$ interval energies $u_\pi \sim u(\nu)$ instead of h , and including the same intermediate equations (12)-(14) as in Planck's derivation [13]. The form of the quantum law is thus reproduced, which was hitherto thought to be inherently quantum mechanical and impossible to derive without the quantum assumptions of harmonic oscillators or boson statistics. The constancy of the equilibrium value of u_π is established as Corollary 3.1 below, also without invoking any nonclassical reasoning, so that u_π may be replaced with h in equation (15) without diminishing the strictly classical character of the present result. \square

Corollary 3.1 (Constancy of wavelength interval energies) *The equilibrium u_π is a thermodynamic constant.*

Proof. By equation (15), the equilibrium u_π does not depend on the total energy, temperature or the geometry of a cavity. If the cavity is enlarged, energy must be added to occupy the additional half-wavelength intervals that become available from the increased volume, in order to keep its temperature steady. To raise the temperature alone, without changing its geometry, energy must be added once again to occupy the higher frequency modes, which would have larger numbers of standing wave $\lambda/2$ intervals, as required by equation (15) for raising the temperature.

For the further conclusion that the equilibrium u_π is the same for all cavities in equilibrium, we must consider the state of equilibrium between pairs of cavities thermally coupled to each other, via a channel for exchanging travelling waves or by contact between their walls, permitting energy exchange indirectly through the wall Doppler shifts. Then, if the cavities are measured to be initially at the same temperature, subsequent thermal interaction between them should cause no change in their total energies or their spectra, as such a change would mean that the net entropy of the two cavities increased upon contact, and hence that they could not have been at the same temperature. Then at each frequency ν , their spectral energy densities $\langle E(\nu, T) \rangle$ should match as well, as a requirement for detailed balance. This would be impossible unless u_π is the same in both, given that T also is the same and all of the remaining factors in equation (15) are constants. The equilibrium value of the $\lambda/2$ interval energy, u_π , is therefore a universal constant, signifying a thermodynamic scale factor of energy for the frequency domain. \square

Proof of Theorem 1. The present analysis has established both that Planck's law implies a classical equipartition (Lemma 1), and that the classical equipartition of confined radiation given by the irreversibility of wall Doppler shifts yields Planck's law (Lemma 3 and Corollary 3.1), as required by the statement of Theorem 1. \square

IV. CONCLUSION

A fundamental if hitherto unanticipated unification of classical and quantum theories has been established above, by rigorous, detailed analysis, evidently for the first time, of the (classical) energies of standing wave modes of radiation in a cavity and of their dynamical (classical) equilibrium under mixing by Doppler shifts due to the thermal motions of the cavity walls. This at first may not seem too surprising to some readers as the radiation equilibrium concerns standing wave modes, which are inherently discrete – the discreteness of standing wave modes is however spatial, and the difficulty all along has been that quantization concerns discretization of energy, which is at best temporal and is not obvious from the spatial discreteness of modes. Many readers will presumably also contend that the present result, being specific to the radiation spectrum, cannot possibly imply that classical physics subsumes all of quantum theory, as claimed in the Introduction – their standpoint would require explaining, however, how the origin and constancy of h can follow completely from classical theory for radiation, yet imply inherently nonclassical physics in other contexts without involving any other constant on a similar fundamental scale as h . Zero point energy and superfluidity theories, for instance, have only h to formally represent their inherent quantumness in the related equations, and Schrödinger's equation would describe a merely classical state space evolution but for the factor $\hbar \equiv h/2\pi$ (cf. [14, §27]) – with h now shown to be a classical constant, akin to Boltzmann's constant k_B , classical explanations for these theories must presumably exist and need to be uncovered. The notion of observer states being fundamentally involved in quantum measurements appears to be new, and may be necessary for the exercise for reasons suggested in the Introduction.

APPENDIX A: ATOMICITY OF WAVELENGTH INTERVALS

Lemma 4 (Wave interval energies) *The energy of a complete wavelength interval of a periodic wave is invariant of its initial phase. The energy of a half-wavelength interval of a sinusoidal wave is invariant of its initial phase.*

Proof. We need to establish, in terms of the functional notation of Section II, that $\partial U(2\pi, \alpha, f)/\partial\alpha = 0$ identically, meaning that the total energy in the volume of the segment remains unchanged even as the wave travels through the volume, and presents different initial phase angles α . This can be done in two parts, first, proving that the α -derivative of $U(2\pi, \alpha, \sin)$, given by equation (2), vanishes, and second, that the α -derivative of the sum $U(2\pi, \alpha, f)$, given by equation (5), vanishes as well. From the elementary calculus of trigonometric functions, we have

$$\frac{\partial}{\partial\alpha} \int_0^\phi \sin^2(\theta + \alpha) d\theta = \frac{\partial}{\partial\alpha} \int_\alpha^{\alpha+\phi} \sin^2 \theta d\theta = \frac{\partial}{\partial\alpha} \left[\frac{\phi}{2} - \frac{\sin(2\alpha + 2\phi) - \sin(2\alpha)}{4} \right] = 0 \iff \phi = n\pi, \quad (\text{A1})$$

which yields $\partial U(2\pi, \alpha, \sin)/\partial\alpha = 0$ on setting $n = 2$. This proves the second statement in the theorem.

The second part of this proof involves a basic property of a Fourier series expansion, that it only contains wavelengths λ_i that are *exact factors* of λ , since each of these components must have the same value at each end of the wavelength segment of f , i.e. $\lambda/\lambda_i = n$, where n is an integer. Therefore,

$$\frac{\partial}{\partial\alpha} U(2\pi, \alpha, f) = \sum_{i=0}^{\infty} F_i \frac{\partial}{\partial\alpha} U\left(\frac{2\pi\lambda}{\lambda_i}, \frac{\alpha\lambda}{\lambda_i}, \sin\right) = \sum_{i=0}^{\infty} F_i \frac{\partial}{\partial\alpha} U(2\pi n_i, \alpha n_i, \sin) = 0 \text{ identically.} \quad (\text{A2})$$

It would be noticed that this result holds in general only for segments of length λ , the full period of the waveform, whereas for sinusoidal functions, and indeed any alternating functions of 50% duty cycle, the constancy would also hold for segments of half the wavelength. This difference is reflected in the fact that equation (A1) has a null for every $\phi = n\pi$, but equation (A2) involves nulls of the component sinusoids only at $\phi = 2\pi n_i$. \square

APPENDIX B: DENSITY OF MODES

Let $k = 2\pi/\lambda$, the wavenumber representation for frequency. Then for j -th mode, with j half-wavelength intervals, $k_j = j\pi/L$, where L = cavity dimension along the wave. Separation between successive modes is then $\delta k = k_{j+1} - k_j = \pi/L$. The number of standing wave modes in the interval Δk is then $\Delta N = \Delta k / \delta k = L\Delta k / \pi$. For the three dimensions of physical space, $\Delta N(k) = V\Delta^3 k / (2\pi)^3$ or

$$dN(\mathbf{k}) = \frac{V d^3 \mathbf{k}}{(2\pi)^3}. \quad (\text{B1})$$

Using $|\mathbf{k}| = \omega/c$, transforming to spherical coordinates, and multiplying by 2 for polarization, this becomes

$$dN = 2 \times \frac{V 4\pi \omega^2 d\omega}{(2\pi)^3 c^3} = V \frac{8\pi \nu^2 d\nu}{c^3}, \quad (\text{B2})$$

yielding the volume density of modes used in equation (15),

$$n(\nu) d\nu \equiv dN(\nu) = \frac{8\pi \nu^2 d\nu}{c^3}. \quad (\text{B3})$$

APPENDIX C: RELATION TO HAMILTON-JACOBI THEORY

The following derivation, which presumably follows the historical development in the early part of the past century, is taken directly from Goldstein's text [5, 10-8]. The classical Hamilton-Jacobi theory predicts propagating wavefronts in a configuration space, described by the evolution of Hamilton's principal and characteristic functions as (equation 10-142, *ibid.*)

$$S(q, P, t) = W(q, P) - Et \quad (\text{C1})$$

where S and W are the principal and characteristic functions, respectively, and E is the energy of the system. Equation (C1) describes contours of S moving like waves, but the waves are not *a priori* sinusoidal. As in electromagnetic wave theory, moving sinusoidal components are obtained by a trial solution of the form $e^{\mathbf{A}(\mathbf{r}) \pm i(\mathbf{k} \cdot \mathbf{r} \pm \omega t)}$ (equations 10-152 through 10-155, *ibid.*) which then becomes an eigenfunction of the wave equation (C1) of amplitude $a(\mathbf{r}) \equiv e^{\mathbf{A}(\mathbf{r})}$. By the first principles of integral calculus, the total solution must be a sum or integral of all such eigenfunctions, i.e.

$$S(q, P, t) = \int a(\mathbf{r}) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d\omega, \quad (\text{C2})$$

where the domain of \mathbf{r} is the configuration space of the coordinates q and momenta P . Equation (C2) simply defines the Fourier transformation of S . The correspondence to quantum theory comes from the eikonal approximation

$$(\nabla L)^2 = n^2 \quad (\text{C3})$$

where L is the phase or optical path length, and n signifies a refractive index. This requires the notion of wave velocity $u = E/|\nabla W|$, and is valid only in the limit of slowly changing n . For a single particle, Hamilton-Jacobi theory yields

$$(\nabla W)^2 = 2m(E - V) \quad (\text{C4})$$

where V is the potential energy and m , the mass. The similarity of equations (C3) and (C4) suggests a correspondence between W and L , or equivalently between $S \equiv W - Et$ and the total phase of the eigenfunction $k(L - ct)$, implying a proportionality of the particle energy to the frequency of the eigenfunction, $E = \hbar\nu \equiv \hbar\omega/2\pi$, where \hbar is simply a constant of proportionality. Using this, the particle motion can be expressed by the evolution equation

$$\psi = \psi_0 e^{i2\pi S/\hbar}, \quad (\text{C5})$$

which would be a solution of the corresponding ‘‘Schrödinger’s equation’’ for the particle,

$$\frac{\hbar^2}{2m} \nabla^2 \psi - V\psi = \frac{\hbar}{i} \frac{\partial \psi}{\partial t} \quad (\text{C6})$$

with $\hbar = \hbar/2\pi$. Indeed, in terms of S , equation (C6) becomes

$$\left[\frac{1}{2m} (\nabla S)^2 + V \right] + \frac{\partial S}{\partial t} = \frac{i\hbar}{2m} \nabla^2 S \quad (\text{C7})$$

which matches the classical Hamilton-Jacobi equation for S iff $\hbar \nabla^2 S \ll (\nabla S)^2$, corresponding to the short wavelength limit at which the potential V varies little over a wavelength. This correspondence is currently interpreted as reducing quantum theory, as represented by equation (C6), to the classical motion given by equation (C5) if \hbar were zero, this being Bohr’s correspondence principle. Goldstein paraphrases this perspective, which evolved over the first half of the past century, by pointing out that Hamilton could not have discovered Schrödinger’s equation because lacking ‘‘experimental authority for the jump’’, he had no reason to believe that ‘‘the value of h was at all different from zero’’.

This idea, including Bohr’s principle, too is fallacious because equation (C5), $\psi = \psi_0 e^{i2\pi(W-Et)/\hbar}$, defines a Fourier component from the decomposition, equation (C2), so that $\hbar \equiv \hbar/2\pi$ is fundamentally *a scale factor for the frequency domain*. Consequently, *zero is the only value not permissible for h* , since the decomposition as such would become undefinable with this choice. It thus follows that *Hamilton already had mathematical authority for $h > 0$* .

The only missing data were the magnitude of h , which had to be experimentally determined, and, more importantly, the thermodynamic significance presented here. It is only this significance that makes the experimentally determined value, of approximately 6.654×10^{-34} J s, special for identifying the Fourier scale factor \hbar with Planck’s constant h , just as a related thermodynamic significance makes the experimental value of 1.3802×10^{-23} J special for k_B .

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- [19] Reducibility of the constancy of the speed of light postulates of special relativity is proved in the appendix of [15]. Current ideas that quantum mechanics is irreducible and *more* fundamental unfortunately place modern physics on a speculative, almost theological foundation. The power of physics lies not in exotic leaps of faith, but in mundane reasoning to understand, describe and otherwise deal with the same empirical data. The present result should thus be seen as advocating and enabling a rigorous understanding of quantum physics, rather than merely questioning its foundation.
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